

Pairing in ultracold Fermi gases in the lowest Landau level

G. Möller¹, Th. Jolicoeur², and N. Regnault³

¹*Cavendish Laboratories, J. J. Thomson Avenue, Cambridge UK, CB3 0HE*

²*Laboratoire de Physique Théorique et Modèles statistiques, Université Paris-Sud, 91405 Orsay, France and*

³*Laboratoire Pierre Aigrain, Département de Physique, ENS, CNRS, 24 rue Lhomond, Paris, F-75005*

(Dated: July 7, 2008)

We study a rapidly rotating gas of unpolarized spin-1/2 ultracold fermions in the two-dimensional regime when all atoms reside in the lowest Landau level. Due to the presence of the spin degree of freedom both s -wave and p -wave interactions are allowed at ultralow temperatures. We investigate the phase diagram of this system as a function of the filling factor in the lowest Landau level and in terms of the ratio between s - and p -wave interaction strengths. We show that the presence of attractive interactions induces a wide regime of phase separation with formation of maximally compact droplets that are either fully polarized or composed of spin-singlets. In the regime with no phase separation, we give evidence for fractional quantum Hall states. Most notably, we find two distinct singlet states at the filling $\nu = 2/3$ for different interactions. One of these states is accounted for by the composite fermion theory, while the other one is a paired state for which we identify two competing descriptions with different topological structure. This paired state may be an Abelian liquid of composite spin-singlet Bose molecules with Laughlin correlations. Alternatively, it may be a known non-Abelian paired state, indicated by good overlaps with the corresponding trial wavefunction. By fine tuning of the scattering lengths it is possible to create the non-Abelian critical Haldane-Rezayi state for $\nu = 1/2$ and the permanent state of Moore and Read for $\nu = 1$. For purely repulsive interactions, we also find evidence for a gapped Halperin state at $\nu = 2/5$.

I. INTRODUCTION

Cold atomic gases are an ideal system for the study of novel quantum phenomena, due to the increasingly versatile experimental techniques available to control these systems. Recently an intense activity focused onto the case of spin-1/2 fermions with attractive interactions. In the realm of ultralow temperatures, the dominant process of interaction is s -wave scattering which is allowed by the Pauli principle when colliding fermions are in a spin singlet state. The strength of this scattering can be tuned through Feshbach resonances by applying a static external magnetic field. On one side of the resonance there is no real bound state close to zero energy while on the other side there are weakly bound diatomic molecules. If we now consider a gas of atoms then the side with no bound state will lead to a BCS instability with formation of a ground state with pairing correlations while the other side will lead to Bose condensation of molecules. The crossover between these two regimes has been the subject of many theoretical condensed-matter studies – atomic physics experiments are now able to probe this important regime. Superfluidity can be observed in this system of fermions by imposing a rotation to the gas. There is then occurrence of quantized vortices arranged in a regular lattice.[1] This is an interesting parallel with atomic Bose gases which also display the Abrikosov lattice of vortices when set in rotation.

In the case of atomic Bose gases it is expected that if the rotation is fast enough, and the confinement along the rotation axis is sufficiently strong, the gas will flatten and reach a two-dimensional regime. There is then formation of Landau energy levels familiar from the quantum mechanics of a particle in a magnetic field. When all bosons reside in the lowest Landau level (LLL) there is formation of fractional quantum Hall (FQH) states of the bosons after melting of the vortex lattice.[2, 3, 4, 5] It is thus a very natural question to investigate what happens under similar circumstances to a gas of spin-1/2 fermions. Rotation of the system translates into two additional forces in the rotating frame : the centrifugal force and the Coriolis force. In a trap with a harmonic confining potential the restoring force is linear in distance from the axis of rotation, as is the centrifugal force. For fast enough rotation these forces may thus nearly compensate each other and we are left with the Coriolis force which is formally equivalent to a static external magnetic field applied on the neutral atoms. Under these conditions, the two-dimensional regime is very special since the kinetic energy is quenched and one-particle energy levels form degenerate sets – the Landau levels responsible for the appearance of the quantum Hall effect in condensed matter physics. At low enough temperatures all particles will occupy the LLL and solely the interactions between them will determine the nature of the ground state : there is no longer any competition between kinetic energy and potential interaction energy. When there are only few bosons per available quantum state in the LLL, correlated liquid phases with special properties emerge, the so-called fractional quantum Hall liquids. For spinless bosons interacting via s -wave scattering, the filling factor 1/2 leads to a ground state which is exactly given by the Laughlin wavefunction.[2] At other filling factors there are other FQH states, some that belong to the standard lore [4] and some more exotic states.[3] If we now consider ultracold spin-1/2 fermions, which may, in principle, reach the same quantum Hall regime. The Zeeman energy that lifts the

degeneracy between the two fermion species can be manipulated but in this paper we focus specifically on the case of zero Zeeman energy. In the context of the quantum Hall physics of electrons, this case can be realized only under very special circumstances such as large external hydrostatic pressure applied to the sample. It is known that even with complete spin degeneracy, the electrons prefer to adopt a fully polarized ground state at least for fractions like $\nu = 1$ and $\nu = 1/3$. Ultracold spin-1/2 fermions have interactions that are of a very different nature compared to electrons in semiconductor heterostructures. Scattering of spin-1/2 fermions at ultralow temperatures is normally dominated by s -wave processes and p -wave scattering is suppressed. However manipulation of Feshbach resonances can be used to boost p -wave scattering up to the same order of magnitude of s -wave interactions in cold gases of potassium [6, 7, 8] as well as lithium.[9, 10, 11] It is thus physically relevant to explore the physics as a function of the ratio of the s - and p -wave scattering lengths.

In this paper, motivated by experimental advances, we investigate the quantum Hall physics of spin-1/2 ultracold fermions. We concentrate on the balanced case with equal populations of both spin states. In the LLL there are now two relevant parameters: the filling factor and the ratio of s -wave and p -wave scattering. We stress that only the ratio is relevant since in the absence of kinetic energy the overall energy scale factors out of the physics. The complete disappearance of kinetic energy in the LLL leads to a wide regime of phase separation if there is attraction between the atoms in some allowed spin channel. The atomic system then prefers to form a maximally compact droplet of spin zero if there is s -wave attraction and a ferromagnetic droplet with maximal spin when there is attraction in the p -wave channel. These states have a very simple explicit form but they do not exhaust all the possibilities for the ground states of the system. To characterize the quantum Hall states we use exact diagonalizations of the many-body problem in the spherical geometry. Candidate quantum Hall liquids have a very definite ratio of flux vs number of particles which depends upon their internal topological order and can be used as an identifying signature. We find evidence for incompressible quantum Hall states at the filling fraction $\nu = 2/3$. If the s -wave interaction is repulsive enough there is formation of a singlet state which can be described by standard composite fermion construction. For weaker interactions in the s -wave channel there is a transition towards a state which we tentatively describe as an Abelian paired state as envisioned by Halperin [12] with singlet molecules forming a standard Bose-Laughlin state at an effective filling factor $\nu_B = 1/6$. However this may not be the whole story since we also find very good overlap with a non-Abelian paired state introduced by Ardonne et al. [13]. At $\nu = 2/5$, we identified a gapped spin-singlet state described by a Halperin wavefunction.[12] Finally we also show that there are critical points, i.e. gapless systems, at filling fractions $\nu = 1, 1/2$ that are described by the non-Abelian Haldane-Rezayi state [14] for $\nu = 1/2$ and the permanent state [15, 16, 17] for $\nu = 1$.

In section II we discuss the peculiarities of ultracold fermions with spin in the LLL in rotating systems. Our results for the various filling factors are exposed in section III. Conclusions are given in section IV.

II. INTERACTING ULTRACOLD FERMIONS WITH SPIN IN THE LLL

We first discuss the one-body problem for a particle trapped in an anisotropic rotating potential. We consider the case when there is strong confinement along the z -axis which is also the rotation axis. The one-body Hamiltonian in the rotating frame can be written as:

$$\mathcal{H}_R = \frac{1}{2M} [\mathbf{p} - M\Omega\hat{\mathbf{z}} \times \mathbf{r}]^2 + \frac{1}{2}M\omega_z^2 z^2 + \frac{1}{2}M(\omega_\perp^2 - \Omega^2)\mathbf{r}_\perp^2. \quad (1)$$

In this equation, M is the mass of the fermion, Ω is the rotation velocity, ω_z is the characteristic trapping frequency along the z -axis, ω_\perp is the trapping frequency in the x - y -plane perpendicular to z , and the coordinates in the x - y -plane are $\mathbf{r}_\perp^2 = x^2 + y^2$. We assume that the dynamics of the system is effectively two-dimensional, with the motion along the z -direction confined to the lowest eigenstate of the harmonic confinement $\phi_z \propto \exp[-z^2/2\ell_z^2]$, with $\ell_z = \sqrt{\hbar/M\omega_z}$. The quantum Hall regime may be recovered when $\Omega \approx \omega_\perp$. The Coriolis force, which is formally equivalent to the Lorentz force, mimics a magnetic field $B = 2M\Omega$. The one-body eigenstates are then given by the two-dimensional Landau levels: they are a set of highly degenerate states with a spacing given by the cyclotron frequency $\hbar\omega_c = 2\hbar\Omega$, their degeneracy being proportional to the area of the system. The eigenfunctions of the LLL are given by:

$$\phi_m(z) = \frac{1}{\sqrt{2^{m+1}\pi m!}} z^m e^{-|z|^2/4\ell_0^2}, \quad (2)$$

where z is the complex coordinate in the x - y -plane, the length scale is set by the “magnetic” length $\ell_0 = \sqrt{\hbar/2M\Omega}$ and m is a *positive* integer. If we are not exactly at the critical rotation velocity there will be a remaining harmonic potential. This residual effect is not expected to affect states that are incompressible i.e. robust to density changes due to a bulk gap. This is the case of the so-called fractional quantum Hall states that we study in this paper.

In this work we focus on the case of two species of fermions. This is the situation that is relevant to the study of the BEC-BCS crossover. We will consider two degenerate species and treat them as spin-1/2 fermions. Here degenerate means that the energy splitting between these states is much smaller than their interaction energy. The “spin” that we use may have a complex microscopic origin. For example, in ${}^6\text{Li}$ in zero magnetic field the low lying states are a hyperfine doublet $F = 1/2$ and a quartet $F = 3/2$. With a moderate field the ground state becomes a triplet and the two low-lying states of this triplet are of particular interest to create stable spin mixtures. Notably it is known that there is a pronounced Feshbach s -wave resonance in the scattering between these two states. In addition there is also a p -wave resonance between these states. By tuning the magnetic field it is thus possible to have some control of the relative scattering strength between s -wave and p -wave interactions. We now show how one can parametrize the interaction Hamiltonian in the LLL. It is well known that the two-body problem in the LLL is trivially solvable for arbitrary rotationally invariant interactions and that the eigenenergies are given by the set of so-called Haldane pseudopotentials [18] V_m :

$$V_m = \frac{\langle \phi_m | V(z) | \phi_m \rangle}{\langle \phi_m | \phi_m \rangle}, \quad m \geq 0, \quad (3)$$

where $V(z)$ is the two-dimensional interaction potential and $\phi(z) \propto z^m \exp(-|z|^2/8\ell_0^2)$ is the relative particle eigenstate with angular momentum m . The even values of m are relevant to the spin singlet channel and the odd values to the spin triplet channel. The s -wave (resp. p -wave) scattering amplitude is related to V_0 (resp. V_1). The interacting N-body problem is then fully defined by the Hamiltonian:

$$\mathcal{H} = \sum_{m \geq 0} V_m \sum_{i < j} \mathcal{P}_{ij}^{(m)}, \quad (4)$$

where $\mathcal{P}_{ij}^{(m)}$ projects the pair of particles i and j onto relative angular momentum m . Neglecting scattering in higher partial waves, the problem only involves V_0 and V_1 given by:

$$V_0 = \sqrt{\frac{2}{\pi}} \frac{\hbar^2}{2M} \frac{a_s}{\ell_0^2 \ell_z}, \quad V_1 = \sqrt{\frac{2}{\pi}} \frac{\hbar^2}{M} \frac{a_p^3}{\ell_0^4 \ell_z}. \quad (5)$$

Here a_s (resp. a_p) is the s -wave (resp. p -wave) scattering length deduced from low-energy scattering limit and we have used explicitly the fact that the wavefunction along z is the Gaussian ground state wavefunction of width ℓ_z . Note there is no explicit spin dependence in the interactions, it is only through the Pauli principle that the spin degrees of freedom are interacting non-trivially. Since one can factor out an overall energy scale the N-body problem in the LLL is thus function only of the ratio V_1/V_0 . It is then convenient to parametrize the interaction Hamiltonian as:

$$\mathcal{H}_\theta = g_0 \cos \theta \sum_{i < j} \mathcal{P}_{ij}^{(0)} + g_0 \sin \theta \sum_{i < j} \mathcal{P}_{ij}^{(1)}, \quad (6)$$

where g_0 is the overall energy scale. The phase diagram can be represented as a circle described by the angular variable θ .

Our strategy is to perform exact diagonalizations of Eq. (6) for a small number of fermions in the spherical geometry. This kind of calculations pioneered by Haldane has proved fruitful to find incompressible quantum Hall states. Technical details can be found in the work of Fano *et al.* [19] It is convenient to switch to the equivalent magnetic language in which vorticity is now magnetic flux. A sphere can be pierced only by an integer number of flux quanta $N_\phi = 2S$ and the Landau levels can be classified according to their orbital angular momentum. The LLL has momentum S , and hence is $2S + 1$ times degenerate and is spanned by functions $u^{S+M} v^{S-M}$, $u = \cos(\theta/2) e^{-i\phi/2}$ and $v = \sin(\theta/2) e^{+i\phi/2}$, where $M = -S \cdots +S$. There are again pseudopotential parameters as in the infinite plane described above and only two of them are relevant in our case:

$$V_0 = \frac{(2S+1)^2}{S(4S+1)} \sqrt{\frac{2}{\pi}} \frac{\hbar^2}{2M} \frac{a_s}{\ell_0^2 \ell_z}, \quad V_1 = \frac{(2S+1)^2}{S(4S-1)} \sqrt{\frac{2}{\pi}} \frac{\hbar^2}{M} \frac{a_p^3}{\ell_0^4 \ell_z}. \quad (7)$$

We consider the family of Hamiltonians \mathcal{H}_θ on the sphere, allowing a single parameter θ . Many-body states on the sphere are classified by their total angular momentum L and their spin S . In order to search for incompressible quantum states we diagonalize the Hamiltonians \mathcal{H}_θ in the LLL for finite systems with even numbers of particles $N = 2N_\uparrow = 2N_\downarrow$, i.e., total spin $S_z = 0$, at different flux N_ϕ . Candidates for incompressible states have full rotational invariance on the sphere and can be distinguished by their angular momentum $L^2 = 0$. Such states form families defined by a specific relationship between flux and number of particles of the form $2S = (1/\nu)N - \sigma$ where the constant

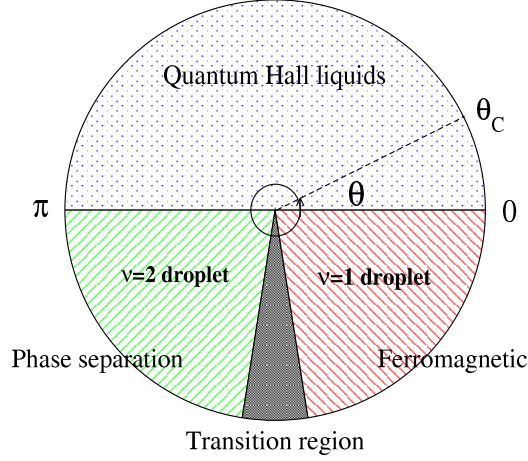


FIG. 1: (color online) Schematic phase diagram for balanced Fermi gases in the LLL: For $0 \lesssim \theta \lesssim \pi$, or $V_1 > 0$, several incompressible quantum liquids can be realized. For $V_1 < 0$, i.e., $-\pi \lesssim \theta \lesssim 0$, the sign of V_0 determines whether the system is unstable to phase separation into a locally $\nu = 2$ droplet ($V_0 < 0$, $-\pi \lesssim \theta \lesssim -\pi/2$) or into a ferromagnetic state with locally $\nu = 1$ ($V_0 > 0$, $-\pi/2 \lesssim \theta \lesssim 0$). The dotted line locates the value $\theta_C \approx \pi/7$ in the phase diagram.

σ is called the *shift*. This shift – which is irrelevant in the thermodynamic limit – is nevertheless a very useful tool to differentiate between states having different internal structures.

We adopt the convention to measure energies in terms of density corrected magnetic lengths $\ell'_0 = \ell_0 \sqrt{2S\nu/N}$. [20] Note that this leads to different finite size scalings for the s - and p -wave channel. We separate the factors determining the finite-size scaling from the coupling constants g_s and g_p setting the scale of interactions in the two scattering channels we consider:

$$V_0 \equiv \frac{(2S+1)^2}{S(4S+1)} \frac{\ell_0'^2}{\ell_0^2} g_s, \quad V_1 \equiv \frac{(2S+1)^2}{S(4S-1)} \frac{\ell_0'^4}{\ell_0^4} g_p. \quad (8)$$

At a given number of particles N and filling factor ν , we then define the interaction Hamiltonian \mathcal{H}_θ by setting $g_s = \cos(\theta)g_0$ and $g_p = \sin(\theta)g_0$, thus defining the overall energy scale g_0 .

III. PHASE SEPARATION

We start our discussion by some general remarks about the phase diagram as a function of θ . The quantum Hall regime in solid state physics, i.e., electrons interacting through the Coulomb potential corresponds to all V_m positive and decreasing with m . This is realized in the upper right quadrant of the circle in Fig. 1. We expect to recover the physics of electronic FQHE with zero Zeeman energy there. Indeed the ratio V_1/V_0 equals $1/2$ for pure Coulomb interaction in the LLL (neglecting the influence of layer width) and this means $\theta = \text{atan}(1/2) \approx 0.46$ close to $\pi/7$.

The possibility of attractive channels may lead to novel features in the remainder of the phase diagram. In the upper left quadrant we have attraction in the s -wave spin-singlet channel competing with repulsive p -wave interactions: hence there will be an interplay between pairing and Laughlin-like correlations.

If both s - and p -wave interactions are attractive, as is the case for $\pi \lesssim \theta \lesssim 3\pi/2$, one has to consider the possibility of phase separation. Indeed in the LLL the kinetic energy is frozen and is not an obstacle to clustering of particles in a state of maximal density favored by attractive interactions. In the lower left quadrant it is thus favorable to make $S=0$ pairs to maximize the V_0 interaction and then to put all pairs as close as possible to maximize V_1 . The state is then simply a Slater determinant constructed by occupation of orbitals of the LLL by singlet pairs. This state is a droplet with maximum local density and its local filling factor is thus $\nu = 2$. The occupation pattern is described in Fig. 2. This phenomenon happens for any number of particles and is always clearly observed in our numerical studies at all filling factors. The ground state is then a state with the maximal possible angular momentum L_{tot} . The value is given by the maximum possible projection of the angular momentum $L_z^{\text{max}} = L_z^{\text{max}}$ where $L_z^{\text{max}} = 2S + 2(S-1) + 2(S-2) + \dots + 2(S+1 - N_e/2)$.

Finally, for $3\pi/2 \lesssim \theta \lesssim 2\pi$, there is also phase separation but now with formation of a fully ferromagnetic droplet as pictured in Fig. 2. The local filling factor is now $\nu = 1$. The maximum value of the momentum is obtained by occupying each orbital exactly once: $L_{\text{tot}} = L_z^{\text{max}}$ where $L_z^{\text{max}} = S + (S-1) + (S-2) + \dots + (S+1 - N_e)$.

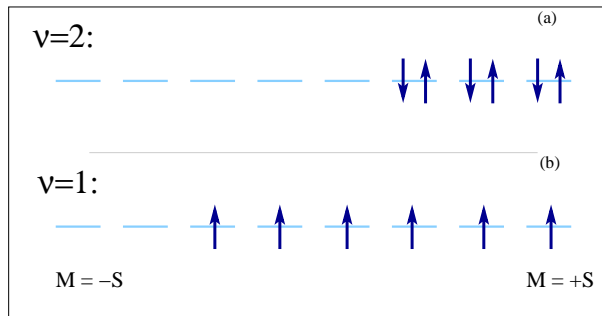


FIG. 2: (color online) The maximum density droplets that can form in a system with attractive interactions. When there is attraction in both s and p wave channels it is best to make a spin singlet droplet of maximum density. This is constructed by filling exactly all orbitals with a singlet pair (a). When there is attraction in the p -wave channel but repulsion in the s -wave channel the maximum density droplet is now fully spin-polarized, i.e. it is a local $\nu = 1$ quantum Hall ferromagnet (b). The state at $S_z = 0$ as studied here is the spin rotation of state (b) into the x - y -plane.

In contrast to incarnations of this state in electronic bilayer systems, there is no charging energy for unbalanced spin-populations in cold atomic gases since these systems are neutral. We conclude that this phase has a strong susceptibility to rotations of the total spin out of the x - y -plane. This ferromagnetic phase supports skyrmion physics which provides opportunities for novel experimental probes of this phenomenology in atomic gases. In our finite system simulations, we find that the transition between the $\nu = 2$ and $\nu = 1$ regimes is not direct, but proceeds via intermediate states. We note also that these two regimes were also found to arise naturally as the limiting two-dimensional behavior of (spin-) density waves of rapidly rotating Fermi gases with s -wave interactions in a three-dimensional regime for $V_0 > 0 (< 0)$. [21]

IV. INCOMPRESSIBLE STATES

Due to the limitation of numerical calculations to a small number of particles, we concentrate on the most prominent filling fractions. With the presence of attraction in the s -wave channel, it is expected that some kind of pairing will play a role. Wavefunctions including pairing have been suggested by Halperin [12] in the context of electronic systems. If two fermions with spin-1/2 form a bound singlet pair, then this fluid of pair will have charge two and thus feels a flux which is twice the original flux. The number of pairs is also half the number of fermions. Now these pairs can have Laughlin-like correlations under appropriate circumstances. They will have then a special flux-number of particle relationship:

$$2 \times 2S = m_B \left(\frac{N}{2} - 1 \right), \quad (9)$$

where pairs form a Laughlin fluid with exponent $m_B = 2, 4, \dots$ with a Bose filling factor $\nu_B = 1/m_B$. The original fermions then form a state with $\nu = 4/m_B$ which is a spin-singlet. The spectral signature of such a state is a singlet ground state and it is natural to expect that the excited states display a well-defined magnetoroton branch resulting from the Laughlin nature of the Bose fluid. This branch should thus extend up to angular momentum $L = N/2$, i.e., to the number of (composite) *bosons*. Such Abelian paired state are not thought to be realized in electronic systems even at zero Zeeman energy. We expect that FQHE states describable by a composite fermion construction are relevant to small values of θ close to the Coulomb value $\theta_C \approx \pi/7$.

In the following section, we discuss the different quantum Hall states that are realized for repulsive p -wave interactions at various filling factors ν . We have searched for systems with a singlet ground state as a function of the number of particles and flux. Due to the exponential growth of the size of the Hilbert space, only a small set of values can be investigated in detail. Strictly speaking, identification of a fractional quantum Hall state requires finding a whole series of states with a definite relation between number of particles and flux with a smooth behavior of physical observables with system size increasing towards the thermodynamic limit. Practically, one should keep in mind that our assignments to quantum Hall fractions are tentative. We are guided by the previously mentioned scheme of Abelian pairing (9) and by the standard composite fermion construction.

A. Filling factor $\nu = 2/3$

We first discuss the possibility of observing the Abelian paired state for $m_B = 6$. The reasoning above leads to a filling factor $2/3$ and a shift equal to -3 . We find strong candidate states for $N = 6, 8, 10$ in wide range of values of θ that includes the so-called hollow-core point $\theta = \pi/2$. In all cases there is a clear gap between a $L = 0, S = 0$ ground state and excited states, the lowest-lying excited states having the expected structure of a magnetoroton branch extending up to $L = N/2$ and having $S = 0$, i.e., they are made of excitations of unbroken singlet pairs. This is most clearly seen in the center of the gapped phase around $\theta \approx \pi/2$. However since there is no available explicit simple wavefunction for the Abelian paired state we cannot perform any overlap calculation. For large values of θ these states are killed by a collapse towards phase separated maximum density droplets. The situation is more interesting for small values of θ . Here we find a transition with a collapse of the gap for $\theta \approx 0.2\pi$ beyond which excited states are nearly degenerate. This transition appears as a rapid crossover as a function of θ . For smaller values of θ we instead observe another series of states with a shift equal to -1 , and this is exactly what we expect from the composite fermion construction which known to be relevant for Coulomb interactions. In the realm of electrons there is a quantum Hall state at $\nu = 2/3$ which is fully spin-polarized and is simply the particle-hole conjugate of the celebrated Laughlin state at $\nu = 1/3$. However it has been observed [22, 23, 24] that reduction of the Zeeman energy leads to the disappearance of this state and formation of a spin-singlet state with the same filling. This state can be explained straightforwardly in the composite fermion construction with spin.[25, 26] In this scheme the CFs exactly fill the effective lowest Landau level with one singlet pair for each available orbital. One has thus $2(|2S^*| + 1) = N$ where $2S^*$ is the reduced flux felt by the CFs. To reach total filling $\nu = 2/3$ this effective flux is negative and given by $2S^* = 2S - 2(N - 1)$. Hence this series of $S=0$ states has $2S = (3/2)N - 1$. For the case of electrons with Coulomb interactions and zero Zeeman energy, it has been shown that CF wavefunctions constructed from this series have extremely good overlap with the numerically obtained ground states (the overlap is 0.998 for $N=8$ and 0.99896 for $N=6$ from ref. 25). We have computed the overlap between the Coulomb ground state and the ground state of the family of Hamiltonians indexed by θ and found that the overlap rises up to 0.9996 for $\theta = 0.025\pi$. It is extremely close to unity in the region where V_0 is larger than V_1 : this means that the physics can be described by the composite fermion scheme.

Concerning the state at shift -3, we note that there is another interesting candidate [13] which is a paired state and also a spin-singlet. Its wavefunction is given by :

$$\Psi_{\text{NASS}} = \text{Pf} \left(\frac{1}{z_i - z_j} \right) \prod_{i < j} (z_i^\uparrow - z_j^\uparrow)^2 \prod_{i < j} (z_i^\downarrow - z_j^\downarrow)^2 \prod_{i,j} (z_i^\uparrow - z_j^\downarrow). \quad (10)$$

In this equation the z_i 's stand for all particle coordinates and $z_i^\uparrow, z_i^\downarrow$ are the coordinates of the two spin components (this notation is a shorthand for the full wavefunction with both spin and space coordinates, it is standard technology to reconstruct the complete state from this notation). The Pfaffian of an antisymmetric matrix is defined as $\text{Pf}(A_{ij}) = \mathcal{A}[A_{12}A_{34} \dots]$ with \mathcal{A} denoting antisymmetrization.

The state (10) is known to exhibit spin-charge separation : the fundamental excitations are spinons with charge zero and spin $1/2$ and spinless holons with charge $\pm 1/3$. The braid statistics of these quasiparticles is non-Abelian; its properties have been investigated by Ardonne et al. [13]. The non-Abelian spin-singlet state Ψ_{NASS} can be constructed as the unique groundstate of an appropriate Hamiltonian composed of (2, 3 and 4-body) hardcore interactions and the total spin S^2 . [This Hamiltonian derives from the corresponding bosonic state $\Psi_b = \Psi_{\text{NASS}}/\Psi_1$, which is obtained as the groundstate of a simple three-body contact interaction with an additional S^2 -term (with Ψ_1 , the wavefunction for a filled Landau level).] We have computed the overlap between the candidate state obtained in this manner and the exact groundstate of our model Hamiltonians. Our results are displayed in the lower panel of Fig. 3, for up to $N=12$ electrons. The overlap is extremely high in the regime where we observe evidence for a gapped state. For $N=14$ at the point $\theta = \pi/2$ the overlap is still as large as 0.71398. It is not clear yet if this non-Abelian state has the same excitation structure that we guessed for Abelian pairing.

The paired character of these states is clear if we look at the correlation function $g(r)$. The Pauli principle requires that for same spin projection this correlation goes to zero for zero separation: $g_{\downarrow\downarrow}(r \rightarrow 0) \rightarrow 0$ and $g_{\uparrow\uparrow}(r \rightarrow 0) \rightarrow 0$. However it does not require that the opposite-spin correlations go to zero. In composite fermion states it is known that $g_{\uparrow\downarrow}$ is very small albeit not zero at $r = 0$. On the contrary when we weaken V_0 we observe a large increase of $g_{\uparrow\downarrow}(0)$ as was first noted by Haldane and Rezayi [14]: see Fig. 5. In this context, we also note that pairing of ultracold fermions was found to be relevant for fermions at filling factor $\nu = 2$ interacting only in the s -wave channel: this system supports a paired state of charge-2 bosons.[27]

Finally we note that there was a related numerical study [28] of electrons at filling $\nu = 2/3$ in the context of bilayer systems for which the layer index plays the role of a pseudospin. The inter and intralayer Coulomb interactions are different from our case. While the study [28] gives evidence for the $\nu = 2/3$ spin-singlet composite fermion state,

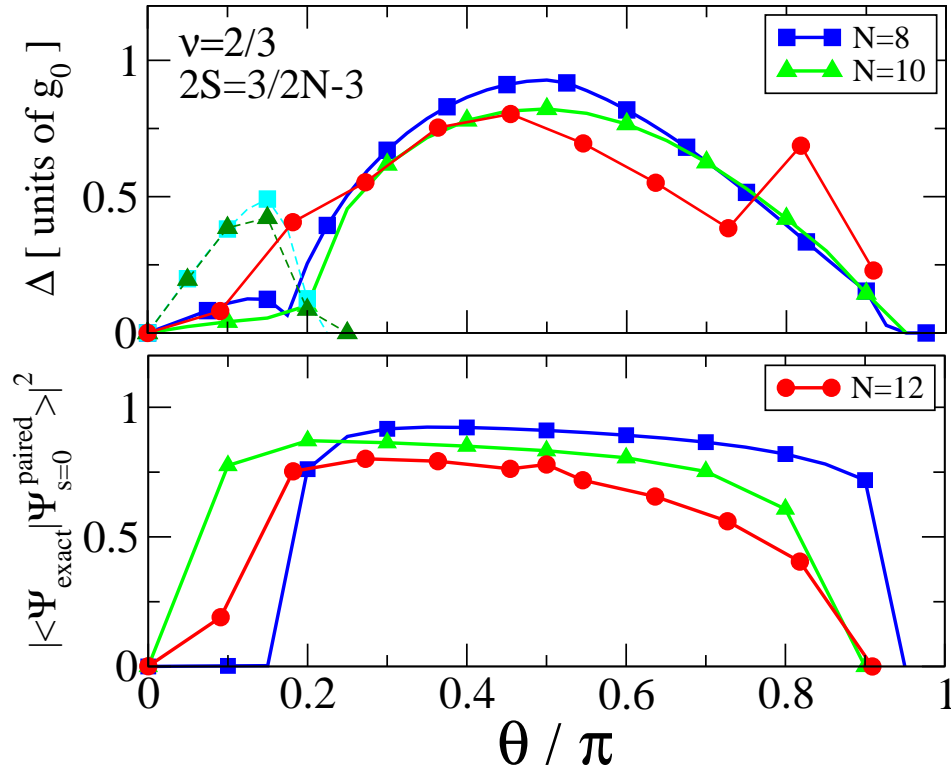


FIG. 3: (color online) Top panel : excitation gaps between the ground state and the first excited state as a function of parameter θ for sizes $N = 8, 10$. The solid lines are computed for the flux $2S = (3/2)N - 3$ as expected for an Abelian paired state. with filling factor $\nu = 2/3$. This state is destroyed for $\theta \lesssim 0.2\pi$ and replaced by the composite fermion state with $S = 0$ and filling $2/3$ with shift $2S = (3/2)N - 1$ as shown with dashed lines. Energies are measured in terms of the overall energy scale g_0 . Lower panel : the overlap between the exact ground state at shift -3 and the non-Abelian state constructed from the Halperin (221) state times a Pfaffian factor. The agreement is very good within the gapped phase

there is also evidence for stability of a decoupled state of type (330) in Halperin notation in some part of the phase diagram. Here, in the ultracold atoms system, we find that this state has always low overlap with the ground state.

B. Filling factor $\nu = 1$

At filling factor unity there is a possibility that the system forms the fully aligned ferromagnetic state with an occupancy probability of one for all orbitals in the LLL. The wavefunction is then given by products of Vandermonde determinant factors:

$$\Psi_{111} = \prod_{i < j} (z_i^\uparrow - z_j^\uparrow) \prod_{i < j} (z_i^\downarrow - z_j^\downarrow) \prod_{i, j} (z_i^\uparrow - z_j^\downarrow), \quad (11)$$

where we have omitted the overall Gaussian factor. The notation Ψ_{111} stands for the powers appearing in the various Jastrow factors and was introduced by Halperin in his work on multicomponent systems.[12] This state, which is the droplet state that always appears in the lowest right-hand part of the phase diagram, has shift $\sigma = 1$ on the sphere. In the zero Zeeman energy limit of the Coulomb problem it is well-known that adding or removing one quantum of flux leads to the formation of a skyrmion [29, 30] which has spin zero. This means that there are spin-0 skyrmion states at flux $2S = N - 2$ in the neighborhood of $\theta = \theta_C$ corresponding to the Coulomb problem.

The Abelian pairing scheme suggests also that there may be a series of incompressible states for $2S = N - 2$ by formation of a Bose Laughlin fluid with $m_B = 4$. However, we do not consistently find a non-zero gap over a wide range of values of θ , as in the case of $\nu = 2/3$ (see Fig. 6). While there is a quite clear gap above a singlet ground state for the systems with $N = 8$ and $N = 14$, the intermediate systems with $N = 10$ and $N = 12$ have a more complicated behavior where the gap drops to zero at $\theta \ll \pi$. We interpret these findings as the apparition of a

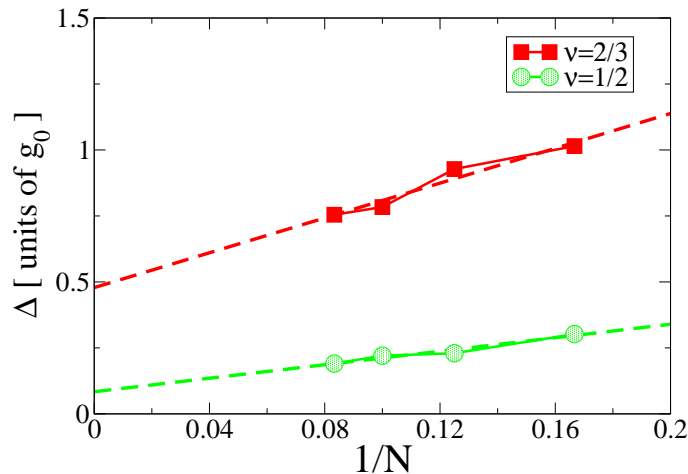


FIG. 4: (color online) Extrapolation of the gaps for the hollow core model ($\theta = \pi/2$, i.e. with only p -wave interactions), expressed in units of the coupling g_0 . The asymptotic values from linear extrapolation over N^{-1} are $\Delta = 0.48(6)$ for $\nu = 2/3$ and compatible with zero for $\nu = 1/2$. The points for $\nu = 2/3$ are taken along the series with shift $\sigma = 3$ as expected from an Abelian paired state. For $\nu = 1/2$ we use the shift of the Haldane-Rezayi state, i.e. $2S = 2N - 4$, which is the maximum density droplet with $\langle E \rangle = 0$. This is evidence for an incompressible fluid at $\nu = 2/3$ and consistent with the Haldane-Rezayi state at $\nu = 1/2$ being gapless.

probably compressible state beyond $\theta \approx 0.17\pi$ which replaces the ferromagnetic droplet state mentioned in section III. However, the data would also be consistent with the existence of an intervening gapped phase in the window $0.17\pi \leq \theta \leq 0.35\pi$. An interesting phenomenon is observed at the point where the transition from the ferromagnetic phase occurs: Read and Rezayi [17] have constructed a wavefunction called the permanent which has exactly the same shift $\sigma = 2$ and is given by:

$$\Psi_{\text{perm}} = \text{per} \left(\frac{1}{z_i^\uparrow - z_j^\downarrow} \right) \Psi_{111}. \quad (12)$$

This state is constructed from a conformal field theory which is non-unitary [15, 16] and is thus expected to be critical.[31] Read and Rezayi have proposed that this state should occur at the phase boundary of a ferromagnet. The relationship with the magnetic instabilities has been explored in more detail in Green's PhD thesis.[32] We have thus computed the overlap between the permanent state Eq. (12) and the ground state of our model Hamiltonian: see Fig. 7 as a function of θ . We find that right at the transition point seen in the gap, the overlap rises to values extremely close to unity: the maximum overlap ranges from 0.998(3) for $N = 8$ to 0.99(1) for $N = 14$. This high overlap we show for the permanent state right where the gap of an adjacent ferromagnetic phases collapses gives numerical support for the idea that the permanent state is right at the phase termination of a ferromagnet. By analogy with the work of D. Green,[32] it is likely that the phase beyond the permanent has helical order, but our small systems do not allow us to analyze this in detail.

C. Filling factor $\nu = 1/2$

The problem of the nature of Coulomb ground state at $\nu = 1/2$ has been of much interest in electronic systems since there is evidence for an incompressible quantum Hall state in the *second* Landau level, at $\nu = 2 + 1/2$. At the present time, the best candidate for describing the FQHE of this state is the non-Abelian Moore-Read state.[15, 33] Historically Haldane and Rezayi [14] introduced a candidate wavefunction which is a spin singlet:

$$\Psi_{\text{HR}} = \det \left(\frac{1}{(z_i^\uparrow - z_j^\downarrow)^2} \right) \prod_{i < j} (z_i - z_j)^2, \quad (13)$$

where the product in the Jastrow factor runs over all particle indices irrespective of the spin projection. The Haldane-Rezayi (HR) state at $\nu = 1/2$ occurs for $N_\phi = 2N - 4$ on the sphere and is the exact ground state of the simplest

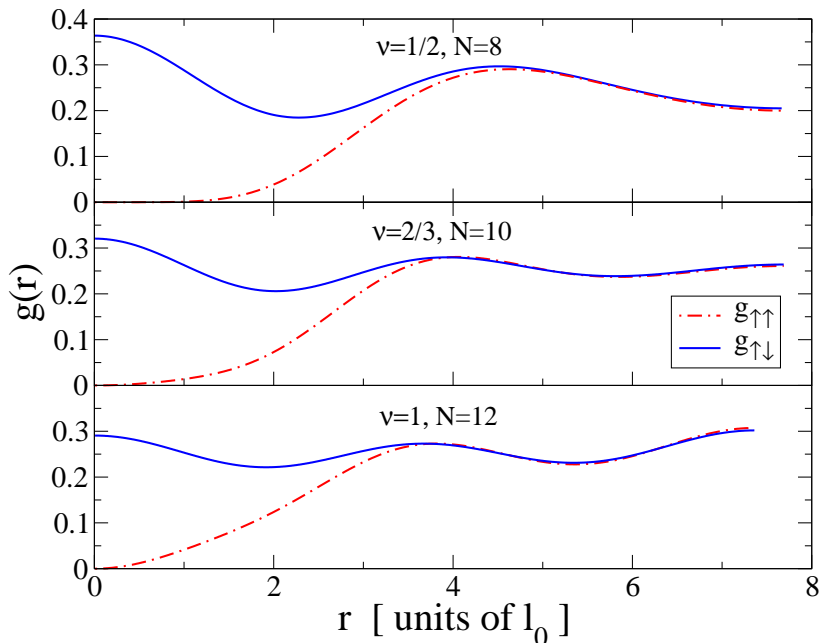


FIG. 5: (color online) Correlation functions as a function of particle separation in units of the magnetic length for the spin-singlet ground states of the hollow-core Hamiltonian $\theta = \pi/2$ (only p -wave scattering). From top to bottom: $\nu = 1/2$ ($N=8$), $\nu = 2/3$ ($N=10$) and $\nu = 1$ ($N=12$). Only the $\nu = 2/3$ state will correspond to a gapped quantum hall state in the thermodynamic limit (probably). In all cases there is a maximum of $g_{\uparrow\downarrow}$ at the origin which is indicative of the paired character of the state.

possible interaction involving both spin species: pure p -wave interactions, also known as the hollow core model (HCM) [14]: in our language this corresponds to $\theta = \pi/2$. Since it can be derived from a non-unitary conformal field theory it is presumably gapless.[31, 34] This is what we find from our numerical studies at $\theta = \pi/2$: see Fig. 4. Extrapolation along this series of states is compatible with zero gap (even though a small but finite gap cannot be excluded). The same shift can also be considered as candidate for Abelian paired states with $m_B = 8$. However, as in the case of $\nu = 1$ we do not find evidence for an extended gapped phase. Neutral gaps are displayed in Fig. 8. The finite-size gap does not peak for pure p -wave interactions as for the other filling factors in this family of states. Rather, there is a two-peak structure which is indicative of the presence of several phases. The knowledge of the HR critical point right at $\theta = \pi/2$ even suggests that there may be an extended critical region but our limited data does not allow a firm conclusion. This is left for future work.

D. Filling factor $\nu = 2/5$

The Hamiltonian \mathcal{H}_θ yields precisely the Halperin wavefunction Ψ_{332} [analogous to Eq. (11)] with shift $N_\phi = 5/2N - 3$ as its exact zero-energy groundstate in the entire sector $0 \leq \theta \leq \pi/2$: For positive pseudopotential coefficients V_0 and V_1 , a zero-energy state of maximum density is obtained if particles with same spin have a relative angular momentum $m_{\text{rel}} \geq 0$ and particles with different spin have $m_{\text{rel}} \geq 1$. The Halperin state Ψ_{332} is the maximum density state with proper symmetries that satisfies these requirements. The nature of its excitations however depends upon θ . For instance, at $N = 10$ we find that the lowest excited state has spin $S = 2$ for $\theta \lesssim 0.3\pi$ while it becomes a spin singlet at larger θ . At the HCM point, the HR state at $\nu = 1/2$ instead becomes the maximum density state with $E = 0$. From the construction of the Halperin state, we expect the gap to be determined approximately by the smallest value of V_0 and V_1 , with the lowest energy excitation breaking either condition on the relative angular momentum. Indeed, numerics show a linear behavior of the gap close to $\theta = 0$ and $\theta = \pi/2$: see Fig. 9. For larger θ , the spectrum at $\nu = 2/5$ is found to be gapless. We conjecture that there are no incompressible states with $\nu \leq 1/2$ for the interval $\pi/2 \leq \theta \leq \pi$.

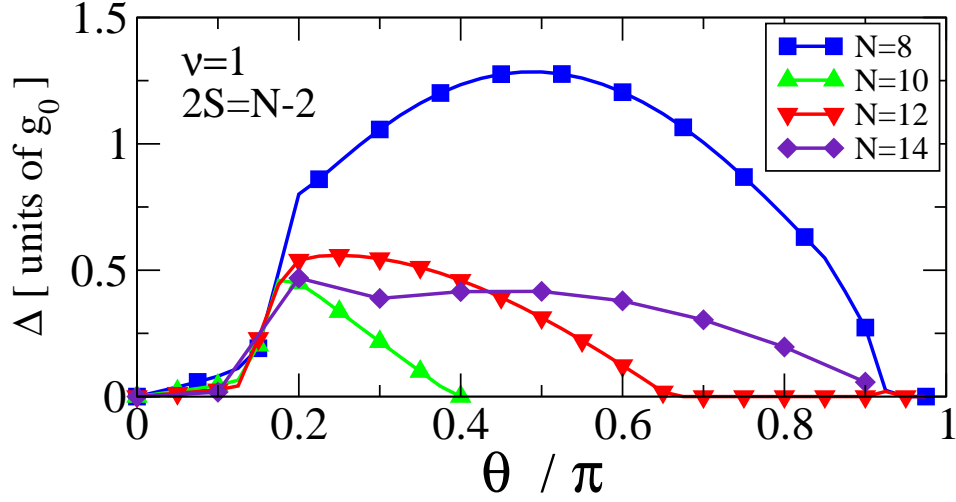


FIG. 6: (color online) Neutral excitation gaps of the spin-singlet states for $2S = N - 2$. While the $N = 8$ and $N = 14$ data suggests the appearance of an incompressible state this conclusion is not supported given the aliasing with smaller or vanishing gaps at $N = 10$ and $N = 12$. Close to $\theta \approx 0.12\pi$ there is a transition towards the ferromagnetic maximally compact state – the groundstate at this transition point, there is some evidence

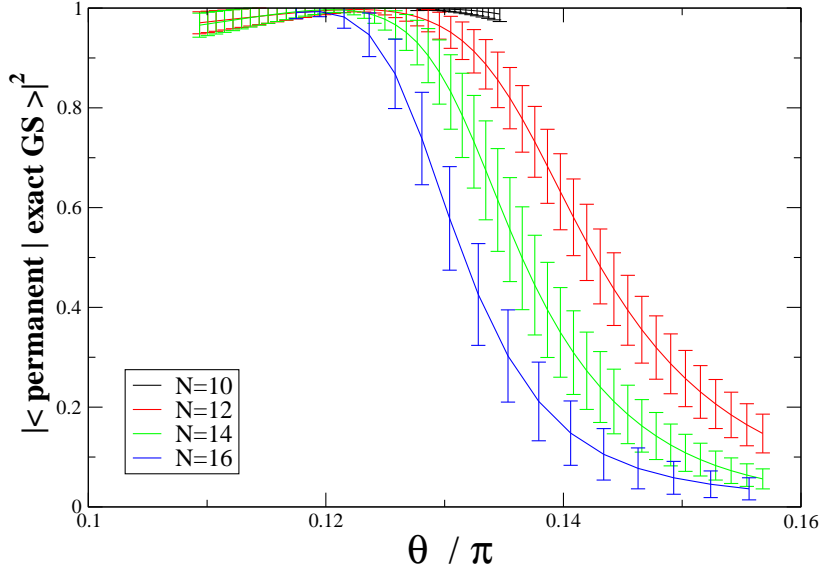


FIG. 7: (color online) The square overlap between the exact ground state for $N = 12, 14, 16$ and the permanent wavefunction. The error bars come from the Monte-Carlo process used to evaluate the scalar product.

V. CONCLUSIONS

We have studied the possible quantum Hall states that may be realized with ultracold fermionic atoms of spin-1/2. This requires first to be able to put all atoms in the LLL for example by rotation of the system at low enough temperatures. Manipulation of the relative strengths of the scattering lengths allow in principle to probe regimes that are inaccessible to electronic systems. When there are attractive interactions in the p -wave channel we find that there is phase separation. The atoms will then form a maximally compact droplet in the presence of the remaining trapping potential. The most interesting case is reached when the p -wave scattering is repulsive and larger than the s -wave scattering. There is then the possibility of paired phases adiabatically connected to a Bose Laughlin fluid of spin singlet pairs of atoms. In principle such phases may be strongly or weakly paired. We have shown that a paired

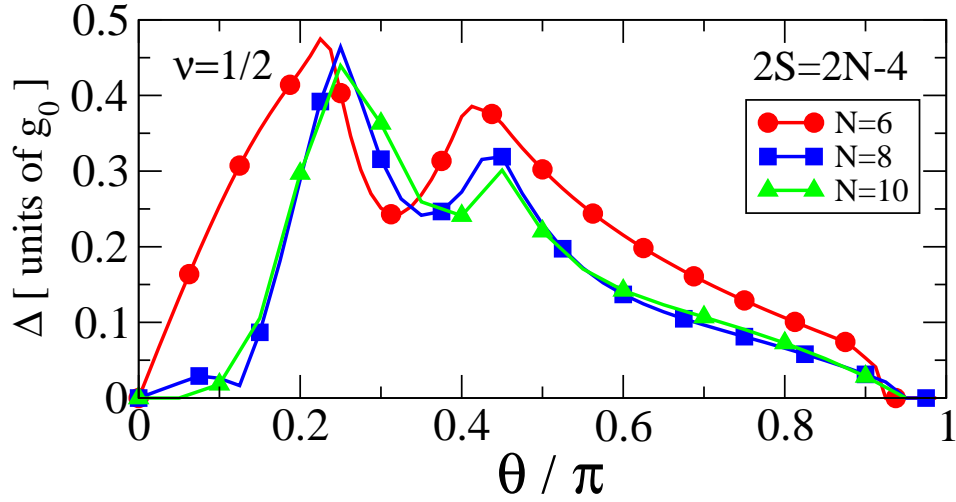


FIG. 8: (color online) Neutral gaps along the Haldane-Rezayi series $2S = 2N - 4$. There is no evidence for a paired incompressible phase.

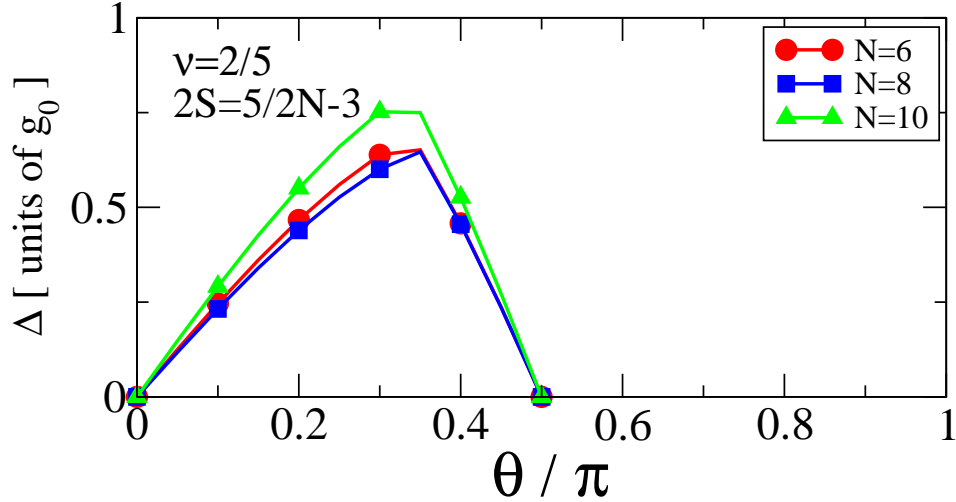


FIG. 9: (color online) The excitation gaps between the ground state and the first excited state as a function of parameter θ for sizes $N = 6, 8, 10$. The solid lines are computed for the flux $2S = (5/2)N - 3$ as expected for the Halperin state Ψ_{332} , with filling factor $\nu = 2/5$. This state exists for $0 \leq \theta \leq \pi/2$ and its gap goes to zero linearly at either extremity of this interval.

state is likely to exist at filling factor $\nu = 2/3$ for an extended range of parameters. When the interaction in the s -wave channel is either weakly repulsive or attractive there is a presumable gapped phase at shift -3 . It has all the spectroscopic signatures we expect from a strongly paired state : there is a magnetoroton branch extending up to $N/2$ values of the angular momentum. We have also shown that this state has a very good overlap with a (weakly) paired non-Abelian spin-singlet state which was derived from the (221) Halperin state in a construction by Ardonne et al. [13].

Fine-tuning the scattering properties allows to reach a region with Coulomb-like interactions where incompressible FQH states are explained readily by the composite fermion picture. There is evidence from our studies for an incompressible state at filling factor $\nu = 2/5$ that is explained by wavefunction originally proposed by Halperin. We have shown also that there are interesting critical states that are already known from the electronic world that can be reached at some special points in our phase diagram. These are the so-called Haldane-Rezayi and permanent state. All these findings give strong motivations to establish and refine experimental techniques that allow to create and manipulate ultracold gases in the LLL regime.

Acknowledgments: One of us (TJ) thank A. H. MacDonald for useful discussions. GM acknowledges fruitful

dialogs with N. R. Cooper and financial support from EPSRC grant GR/S61263/01 and ESF short-visit grant no. 2129 under the INSTANS program. We thank IDRIS for support and computer time on the large scale computation facility Zahir. This work is supported in part by “Institut Francilien des atomes froids” (IFRAF).

-
- [1] M. W. Zwierlein, J. R. Abo-Shaeer, A. Schirotzek, C. H. Schunck, and W. Ketterle, *Nature* **435**, 1047 (2005).
 - [2] N. K. Wilkin and J. M. F. Gunn, *Phys. Rev. Lett.* **84**, 6 (2000).
 - [3] N. R. Cooper, N. K. Wilkin, and J. M. F. Gunn, *Phys. Rev. Lett.* **87**, 120405 (2001).
 - [4] N. Regnault and T. Jolicoeur, *Phys. Rev. Lett.* **91**, 030402 (2003).
 - [5] N. Regnault and T. Jolicoeur, *Phys. Rev. B* **70**, 241307(R) (2004).
 - [6] C. A. Regal, C. Ticknor, J. L. Bohn, and D. S. Jin, *Phys. Rev. Lett.* **90**, 053201 (2003).
 - [7] C. Ticknor, C. A. Regal, D. S. Jin, and J. L. Bohn, *Phys. Rev. A* **69**, 042712 (2004).
 - [8] K. Günter, T. Stöferle, H. Moritz, M. Köhl, and T. Esslinger, *Phys. Rev. Lett.* **95**, 230401 (2005).
 - [9] J. Zhang, E. G. M. van Kempen, T. Bourdel, L. Khaykovich, J. Cubizolles, F. Chevy, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, *Phys. Rev. A* **70**, 030702(R) (2004).
 - [10] C. H. Schunck, M. W. Zwierlein, C. A. Stan, S. M. F. Raupach, W. Ketterle, A. Simoni, E. Tiesinga, C. J. Williams, and P. S. Julienne, *Phys. Rev. A* **71**, 045601 (2005).
 - [11] F. Chevy, E. G. M. van Kempen, T. Bourdel, J. Zhang, L. Khaykovich, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, *Phys. Rev. A* **71**, 062710 (2005).
 - [12] B. I. Halperin, *Helv. Phys. Acta* **56**, 75 (1983).
 - [13] E. Ardonne, F. J. M. van Lankvelt, A. W. W. Ludwig, and K. Schoutens, *Phys. Rev. B* **65**, 041305(R) (2002).
 - [14] F. D. M. Haldane and E. H. Rezayi, *Phys. Rev. Lett.* **60**, 956 (1988).
 - [15] G. Moore and N. Read, *Nucl. Phys.* **B360**, 362 (1991).
 - [16] N. Read and G. Moore, *Prog. Theor. Phys. (Kyoto) Suppl.* **107**, 157 (1992).
 - [17] N. Read and E. H. Rezayi, *Phys. Rev. B* **54**, 16864 (1996).
 - [18] F. D. M. Haldane, *Phys. Rev. Lett.* **51**, 605 (1983).
 - [19] G. Fano, F. Ortolani, and E. Colombo, *Phys. Rev. B* **34**, 2670 (1986).
 - [20] R. Morf, N. d’Ambrumenil, and B. I. Halperin, *Phys. Rev. B* **34**, 3037 (1986).
 - [21] G. Möller and N. R. Cooper, *Phys. Rev. Lett.* **99**, 190409 (2007).
 - [22] J. P. Eisenstein, H. L. Stormer, L. N. Pfeiffer, and K. W. West, *Phys. Rev. Lett.* **62**, 1540 (1989).
 - [23] J. P. Eisenstein, H. L. Stormer, L. N. Pfeiffer, and K. W. West, *Phys. Rev. B* **41**, 7910 (1990).
 - [24] L. W. Engel, S. W. Hwang, T. Sajoto, D. C. Tsui, and M. Shayegan, *Phys. Rev. B* **45**, 3418 (1992).
 - [25] X. G. Wu, G. Dev, and J. K. Jain, *Phys. Rev. Lett.* **71**, 153 (1993).
 - [26] X. G. Wu and J. K. Jain, *Phys. Rev. B* **49**, 7515 (1994).
 - [27] K. Yang and H. Zhai, *Phys. Rev. Lett.* **100**, 030404 (2008).
 - [28] I. A. McDonald and F. D. M. Haldane, *Phys. Rev. B* **53**, 15845 (1996).
 - [29] E. H. Rezayi, *Phys. Rev. B* **36**, 5454 (1987).
 - [30] E. H. Rezayi, *Phys. Rev. B* **43**, 5944 (1991).
 - [31] N. Read, “Conformal Invariance of chiral edge theories”, e-print arXiv:0711.0543 (2007).
 - [32] D. Green, “Strongly correlated states in low dimensions”, e-print arXiv:cond-mat/0202455 (2002).
 - [33] G. Möller and S. H. Simon, *Phys. Rev. B* **77**, 075319 (2008).
 - [34] N. Read and D. Green, *Phys. Rev. B* **61**, 10267 (2000).